Close-to-Optimal Partial Decode-and-Forward Rate in the MIMO Relay Channel via Convex Programming

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***Abstract*—We study the optimization of the data rate achievable with partial decode-and-forward in the Gaussian MIMO relay channel and propose a new algorithm to find close-to-optimal input covariance matrices. Although the optimization is a non-convex problem in its original formulation, we can show that a reformulated version with slightly modified constraint set is convex. In particular, we replace a positive-semidefiniteness constraint by a strict positive-definiteness constraint. This may introduce an inaccuracy, but we conjecture this inaccuracy to be small so that the obtained solutions are close to the global optimal ones.**

I. Introduction

The concept of relay networks was introduced by [1] and is of great interest in recent research as the usage of a relay can increase the achievable data rate compared to the direct transmission. However, neither the capacity of a relay channel nor the optimal strategy of the relay are known so far. In [2], several strategies such as decode-and-forward and compress-and-forward were proposed. In this work, we consider the partial decode-and-foward (PDF) scheme (e.g., [3], [4, Section 9.4.1]), which is an extension of decode-and-forward.

For the PDF scheme, circularly symmetric Gaussian signals have been shown to be the optimal input distribution [5], [6]. Although there are several suboptimal approaches to maximize the achievable PDF rate (e.g., [7]) and solutions for special cases (e.g., [8]), there is still no way to find the optimal transmit covariance matrices for the PDF scheme in the general case. Thus, it is not clear whether the gap between existing solution methods and the cut-set upper bound [2] is due to the suboptimal choice of the covariance matrices or rather inherent to the PDF scheme. To answer this question, a globally optimal solution of the PDF rate maximization is needed. In this work, we take an important step towards finding such a solution.

The PDF rate maximization problem is non-convex, but we can decompose the problem into an outer maximization over a so-called *innovation covariance matrix* $\boldsymbol{C}$ (cf. [6]) and an inner problem to optimize the remaining parameters for a fixed innovation covariance matrix. By restricting the innovation covariance matrix to be strictly positive-definite (i.e., all eigenvalues have to be greater than or equal to a small positive constant ε instead of being greater than or equal to zero), we obtain an approximated problem, for which we can show that both the inner and outer problems can be solved in a globally optimal manner by means of convex programming. Even though the approximation introduces an inaccuracy in scenarios where the optimal distribution requires a rank-deficient innovation covariance matrix, we conjecture the error to be small when choosing a sufficiently small ε.

To obtain a convex reformulation of the rate optimization, we exploit that an arising subproblem is mathematically equivalent to the maximization of the dirty paper coding sum rate [9] in a broadcast channel (BC) with a shaping constraint based on the innovation covariance matrix [6]. This problem can be transformed into a convex minimax problem in a dual multiple access channel (MAC) [10], [11].

In the proposed algorithm, we solve this subproblem by an alternating gradient-projection method (cf., e.g., [12]). For the outer problem of finding the optimal innovation covariance, we use the cutting plane algorithm [13, Section 6.3.3].

II. System Model

The Gaussian MIMO relay channel consists of a source S with N_S transmit antennas, a destination D with N_D receive antennas, and a relay R with N_R antennas. The source transmits data to the destination over a direct channel and with the help of the relay. The channel matrices of the links source-destination, source-relay, and relay-destination are given by $\boldsymbol{H}_\text{SD} \in \mathbb{C}^{N_\text{D}\times N_\text{S}}$, $\boldsymbol{H}_\text{SR} \in \mathbb{C}^{N_\text{R}\times N_\text{S}}$, and $\boldsymbol{H}_\text{RD} \in \mathbb{C}^{N_\text{D}\times N_\text{R}}$, respectively. Perfect channel knowledge is assumed.

A. Partial Decode-and-Forward

Using the partial decode-and-forward strategy [4, Section 9.4.1], the transmit signal $\boldsymbol{x}_\text{S}$ of the source is a superposition of two independent parts $\boldsymbol{u}$ and $\boldsymbol{v}$, where $\boldsymbol{u}$ denotes the part that is sent in cooperation with the relay, and $\boldsymbol{v}$ denotes the part that is directly transmitted without the help of the relay and causes interference at the relay. As proposed in [6], the cooperative part $\boldsymbol{u}$ can be further decomposed into a part $\boldsymbol{q}$ being independent of the relay transmit signal $\boldsymbol{x}_\text{R}$ and a part $\boldsymbol{z}$ being linearly dependent of the relay transmit signal:

$$\boldsymbol{x}_\text{S} = \boldsymbol{u} + \boldsymbol{v} = \boldsymbol{A}\boldsymbol{x}_\text{R} + \boldsymbol{q} + \boldsymbol{v} = \boldsymbol{z} + \boldsymbol{q} + \boldsymbol{v}. \tag{1}$$

As $\boldsymbol{z}$ has linear dependence with the relay transmit signal, and the relay can, due to causality, only transmit data it has previously received,[1] $\boldsymbol{z}$ does not contain new information. The remaining parts $\boldsymbol{q}$ and $\boldsymbol{v}$ containing new information are then called *innovation*, and the covariance matrix $\boldsymbol{C} = \boldsymbol{C}_{\boldsymbol{v}} + \boldsymbol{C}_{\boldsymbol{q}}$ is called *innovation covariance matrix* [6], where $\boldsymbol{C}_{\boldsymbol{v}}$ and $\boldsymbol{C}_{\boldsymbol{q}}$ denote the covariance matrices of $\boldsymbol{v}$ and $\boldsymbol{q}$, respectively.

[1]Note that the rate expressions given in Section II-B are achievable using a block-Markov coding scheme [4, Section 9.4.1].

B. Achievable Data Rates

The achievable data rate with the partial decode-and-forward scheme and circularly symmetric Gaussian signals is given as the minimum of two mutual information expressions

$$R = \min\{R_{\mathrm{A}}, R_{\mathrm{B}}\} \tag{2}$$

[4, Section 9.4.1], where R_{A} and R_{B} can be expressed as [6]

$$\begin{aligned} R_{\mathrm{A}} &= \log_2 \det(\mathbf{I}_{N_{\mathrm{D}}} + \boldsymbol{H}_{\mathrm{SD}} \boldsymbol{C}_{\boldsymbol{v}} \boldsymbol{H}_{\mathrm{SD}}^{\mathrm{H}}) \\ &\quad + \log_2 \frac{\det(\mathbf{I}_{N_{\mathrm{R}}} + \boldsymbol{H}_{\mathrm{SR}}(\boldsymbol{C}_{\boldsymbol{v}} + \boldsymbol{C}_{\boldsymbol{q}})\boldsymbol{H}_{\mathrm{SR}}^{\mathrm{H}})}{\det(\mathbf{I}_{N_{\mathrm{R}}} + \boldsymbol{H}_{\mathrm{SR}} \boldsymbol{C}_{\boldsymbol{v}} \boldsymbol{H}_{\mathrm{SR}}^{\mathrm{H}})} \end{aligned} \tag{3}$$

$$R_{\mathrm{B}} = \log_2 \det(\mathbf{I}_{N_{\mathrm{D}}} + \boldsymbol{H}_{\mathrm{SD}}(\boldsymbol{C}_{\boldsymbol{v}} + \boldsymbol{C}_{\boldsymbol{q}})\boldsymbol{H}_{\mathrm{SD}}^{\mathrm{H}} + \boldsymbol{H}\boldsymbol{R}\boldsymbol{H}^{\mathrm{H}}). \tag{4}$$

We have assumed the noise covariances $\boldsymbol{C}_{\boldsymbol{\eta}_{\mathrm{R}}} = \mathbf{I}_{N_{\mathrm{R}}}$ and $\boldsymbol{C}_{\boldsymbol{\eta}_{\mathrm{D}}} = \mathbf{I}_{N_{\mathrm{D}}}$ w.l.o.g., and we use the joint channel matrix

$$\boldsymbol{H} = \begin{bmatrix} \boldsymbol{H}_{\mathrm{SD}} & \boldsymbol{H}_{\mathrm{RD}} \end{bmatrix}. \tag{5}$$

The joint covariance matrix of $\boldsymbol{z}$ and $\boldsymbol{x}_{\mathrm{R}}$ is denoted by

$$\boldsymbol{R} = \boldsymbol{C}_{\left[\substack{\boldsymbol{z} \\ \boldsymbol{x}_{\mathrm{R}}}\right]} = \mathrm{E}\left[\begin{bmatrix} \boldsymbol{z} \\ \boldsymbol{x}_{\mathrm{R}} \end{bmatrix} \begin{bmatrix} \boldsymbol{z} \\ \boldsymbol{x}_{\mathrm{R}} \end{bmatrix}^{\mathrm{H}}\right]. \tag{6}$$

C. Problem Formulation

We aim at maximizing the achievable data rate under the power constraints P_{S} and P_{R} on the transmit power of the source and the transmit power of the relay, respectively. The power constraints can be formulated as

$$\mathrm{E}\left[\|\boldsymbol{x}_{\mathrm{S}}\|_2^2\right] = \mathrm{tr}(\boldsymbol{C}_{\boldsymbol{v}} + \boldsymbol{C}_{\boldsymbol{q}} + \boldsymbol{D}_{\mathrm{S}}\boldsymbol{R}\boldsymbol{D}_{\mathrm{S}}^{\mathrm{H}}) \leq P_{\mathrm{S}} \tag{7}$$

$$\mathrm{E}\left[\|\boldsymbol{x}_{\mathrm{R}}\|_2^2\right] = \mathrm{tr}(\boldsymbol{D}_{\mathrm{R}}\boldsymbol{R}\boldsymbol{D}_{\mathrm{R}}^{\mathrm{H}}) \leq P_{\mathrm{R}} \tag{8}$$

with the selection matrices

$$\boldsymbol{D}_{\mathrm{S}} = \begin{bmatrix} \mathbf{I}_{N_{\mathrm{S}}} & \mathbf{0}_{N_{\mathrm{S}} \times N_{\mathrm{R}}} \end{bmatrix} \text{ and } \boldsymbol{D}_{\mathrm{R}} = \begin{bmatrix} \mathbf{0}_{N_{\mathrm{R}} \times N_{\mathrm{S}}} & \mathbf{I}_{N_{\mathrm{R}}} \end{bmatrix}. \tag{9}$$

The optimization problem we consider is then given by

$$\begin{aligned} \max_{\substack{\boldsymbol{C}_{\boldsymbol{v}} \succeq \mathbf{0}, \boldsymbol{C}_{\boldsymbol{q}} \succeq \mathbf{0} \\ \boldsymbol{R} \succeq \mathbf{0}}} \ & \min\left\{R_A(\boldsymbol{C}_{\boldsymbol{v}}, \boldsymbol{C}_{\boldsymbol{q}}), R_B(\boldsymbol{C}_{\boldsymbol{v}}, \boldsymbol{C}_{\boldsymbol{q}}, \boldsymbol{R})\right\} \\ \text{s.t.} \quad & \mathrm{tr}(\boldsymbol{C}_{\boldsymbol{v}} + \boldsymbol{C}_{\boldsymbol{q}}) + \mathrm{tr}(\boldsymbol{D}_{\mathrm{S}}\boldsymbol{R}\boldsymbol{D}_{\mathrm{S}}^{\mathrm{H}}) \leq P_{\mathrm{S}} \\ & \mathrm{tr}(\boldsymbol{D}_{\mathrm{R}}\boldsymbol{R}\boldsymbol{D}_{\mathrm{R}}^{\mathrm{H}}) \leq P_{\mathrm{R}} \end{aligned} \tag{10}$$

where the notation $\boldsymbol{A} \succeq \mathbf{0}$ is defined in the sense of positive-semidefiniteness.

III. Primal Decomposition

The optimization problem (10) is non-convex in its original form. However, we can reformulate the problem and show the reformulated version to be convex, as long as the innovation covariance matrix is strictly positive-definite. To do so, we apply the concept of primal decomposition [14] to (10) with the innovation covariance matrix $\boldsymbol{C}$ and the matrix $\boldsymbol{R}$ as coupling variables. A similar approach was pursued in [6] as part of a proof, but has not yet been considered for algorithm design. We obtain the optimization problem

$$\begin{aligned} \max_{\boldsymbol{C} \succeq \mathbf{0}, \boldsymbol{R} \succeq \mathbf{0}} \ & \min\left\{R_A^\star(\boldsymbol{C}), R_B(\boldsymbol{C}, \boldsymbol{R})\right\} \\ \text{s.t.} \quad & \mathrm{tr}(\boldsymbol{C}) + \mathrm{tr}(\boldsymbol{D}_{\mathrm{S}}\boldsymbol{R}\boldsymbol{D}_{\mathrm{S}}^{\mathrm{H}}) \leq P_{\mathrm{S}} \\ & \mathrm{tr}(\boldsymbol{D}_{\mathrm{R}}\boldsymbol{R}\boldsymbol{D}_{\mathrm{R}}^{\mathrm{H}}) \leq P_{\mathrm{R}} \end{aligned} \tag{11}$$

with $R_{\mathrm{B}}(\boldsymbol{C}, \boldsymbol{R}) = \log_2 \det(\mathbf{I}_{N_{\mathrm{D}}} + \boldsymbol{H}_{\mathrm{SD}}\boldsymbol{C}\boldsymbol{H}_{\mathrm{SD}}^{\mathrm{H}} + \boldsymbol{H}\boldsymbol{R}\boldsymbol{H}^{\mathrm{H}})$ and

$$R_A^\star(\boldsymbol{C}) = \max_{\boldsymbol{C}_{\boldsymbol{v}} \succeq \mathbf{0}, \boldsymbol{C}_{\boldsymbol{q}} \succeq \mathbf{0}} \quad R_A \quad \text{s.t.} \quad \boldsymbol{C}_{\boldsymbol{v}} + \boldsymbol{C}_{\boldsymbol{q}} \preceq \boldsymbol{C}. \tag{12}$$

Note that (12) is mathematically equivalent to the dirty paper coding sum rate [9] maximization in a broadcast channel with two users and with a shaping constraint based on the innovation covariance matrix (cf. [6]). The problem of evaluating $R_{\mathrm{A}}^\star$ for given $\boldsymbol{C}$ is discussed in the next section. For $R_{\mathrm{A}}^\star(\boldsymbol{C})$ as a function of $\boldsymbol{C}$, the following theorem holds.

Theorem 1. *For a strictly positive-definite innovation covariance $\boldsymbol{C}$, the expression $R_A^\star(\boldsymbol{C})$ from* (12) *is concave in $\boldsymbol{C}$, and a (concave) subgradient is given by the optimal Lagrangian multiplier for the shaping constraint $\boldsymbol{C}_{\boldsymbol{v}} + \boldsymbol{C}_{\boldsymbol{q}} \preceq \boldsymbol{C}$.*

The proof (included in the full paper) is based on a sensitivity analysis [15, Section 5.6] and on the fact that the BC rate maximization can be transformed into a convex problem with zero duality gap [10], [11].

Corollary 1. *Let $\varepsilon > 0$ and add the constraint $\boldsymbol{C} \succeq \varepsilon\mathbf{I}$ to* (11). *Then, the resulting optimization problem is a convex program.*

Proof: The pointwise minimum of concave functions is concave [15, Section 3.2.3], R_{B} is jointly concave in $\boldsymbol{C}$ and $\boldsymbol{R}$, and $R_{\mathrm{A}}^\star$ is concave for $\boldsymbol{C} \succeq \varepsilon\mathbf{I}$, cf. Theorem 1. ∎

IV. Inner Problem: Broadcast Sum Rate Maximization with Shaping Constraint

Although the broadcast sum rate maximization (12) is a non-convex problem, it can be solved by transforming it into a convex minimax problem in the multiple access channel. According to the duality framework presented in [10], [11], the BC problem with shaping constraint corresponds to a MAC problem with a worst-case noise optimization, and we obtain

$$\min_{\substack{\boldsymbol{C}_{\boldsymbol{\eta}} \succ \mathbf{0} \\ \mathrm{tr}(\boldsymbol{C}_{\boldsymbol{\eta}}) = \sum_k M_k}} \quad \max_{\substack{(\boldsymbol{\Sigma}_k \succeq \mathbf{0})_{\forall k} \\ \sum_k \mathrm{tr}(\boldsymbol{\Sigma}_k) = \sum_k M_k}} \quad R_{\mathrm{MAC}}((\boldsymbol{\Sigma}_k)_{\forall k}, \boldsymbol{C}_{\boldsymbol{\eta}}) \tag{13}$$

where the MAC rate R_{MAC} is concave in the transmit covariance matrices $\boldsymbol{\Sigma}_k$ of the individual users k and convex in the noise covariance matrix $\boldsymbol{C}_{\boldsymbol{\eta}}$ at the receiver. Both the transmit covariance matrices and the noise covariance matrix are subject to a power constraint. Note that R_{MAC} depends on the shaping matrix $\boldsymbol{C}$ via the the uplink channel matrices, which are a function of $\boldsymbol{C}$ (see derivation in the full paper).

V. Proposed Algorithm

For finding the optimal $\boldsymbol{C}$ and $\boldsymbol{R}$ in (11) with the additional constraint $\boldsymbol{C} \succeq \varepsilon\mathbf{I}$, we use the cutting plane algorithm [13, Section 6.3.3]. The algorithm successively refines linear approximations of a concave function. To obtain such approximations, we can use the subgradient of $R_A^\star(\boldsymbol{C})$ derived in Theorem 1 and the gradient of $R_B(\boldsymbol{C}, \boldsymbol{R})$, which can be calculated explicitly (see full paper).

To solve subproblem (13) for fixed $\boldsymbol{C}$ (dependence on $\boldsymbol{C}$ via the uplink channel matrices), we use an alternating gradient projection algorithm as proposed by [12]. This means, we

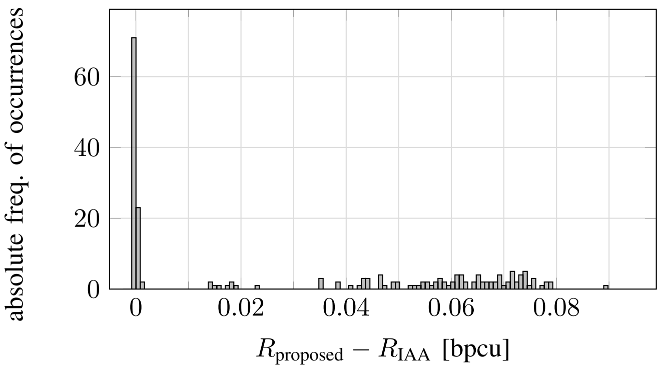

Fig. 1: Histogram of rate gain over IAA [7] for $N_{\text{S}} = N_{\text{R}} = N_{\text{D}} = 2$, $P_{\text{S}} = 100$, $P_{\text{R}} = 10$, $d = 0.8$ and $\varepsilon = 10^{-5} P_{\text{S}}$.

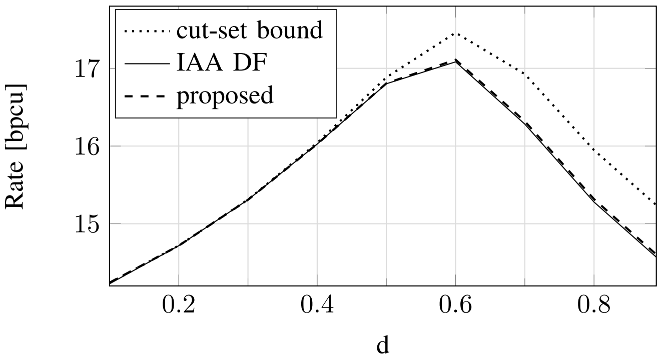

Fig. 2: Average rate compared to IAA [7] and cut-set bound for $N_{\text{S}} = N_{\text{R}} = N_{\text{D}} = 2$, $P_{\text{S}} = 100$, $P_{\text{R}} = 10$, and $\varepsilon = 10^{-5} P_{\text{S}}$.

perform gradient steps and projections onto the constraint set for the inner rate maximization and for the worst-case noise optimization in an alternating manner until convergence.

VI. Results and Conclusion

To evaluate the performance of the proposed algorithm, we compare the results to the inner approximation approach (IAA) from [7] and to the cut-set bound. As in [7], we assume a line network, where the relay lies on a line between source and destination. The distances source-relay, relay-destination, and source-destination are given by $d_{\text{SR}} = d, d \in (0, 1)$, $d_{\text{RD}} = 1 - d$, and $d_{\text{SD}} = 1$, respectively. The channel matrices are given by $\boldsymbol{H}_{\text{AB}} = d_{\text{AB}}^{-\gamma/2} \tilde{\boldsymbol{H}}_{\text{AB}}$ with $\gamma = 4$ and $\text{A}, \text{B} \in \{\text{S}, \text{R}, \text{D}\}$. The individual elements of each $\tilde{\boldsymbol{H}}_{\text{AB}}$ are independent and complex Gaussian distributed with zero mean and unit variance.

The histogram in Figure 1 shows the difference $R_{\text{proposed}} - R_{\text{IAA}}$ for 200 i.i.d. channel realizations with two antennas at each terminal and distance parameter $d = 0.8$. It can be seen that the IAA and the proposed algorithm converge to the same value in many cases. However, there are also cases in which the proposed algorithm achieves a higher rate, meaning that the local optimum found by the IAA method is not the global one in these cases. Figure 2 shows the results for the same scenario with various values of d. By using the proposed method as a benchmark, we can conclude that the IAA has a close-to-optimal perfomance on average, which had not been clear in the first place since the IAA is only a locally optimal method. On the other hand, we can observe that the gap to the cut-set bound (as seen in Figure 2 for $d \geq 0.5$) cannot be closed with the proposed algorithm.

Unlike existing suboptimal approaches to solve the PDF rate maximization problem, the proposed algorithm is not a local approach. Instead, we solve a slightly modified problem in a globally optimal manner. Thus, the proposed algorithm finds the globally optimal solution in cases where the optimal innovation covariance matrix has full rank, and we conjecture that it finds a close-to-optimal solution in the other cases if a sufficient small ε is chosen. If this conjecture holds, the results reveal that the gap to the cut-set bound in Figure 2 is not due to the potentially suboptimal choice of the covariance matrices in existing algorithms, but inherent to the PDF scheme or inherent to the fact that the CSB might not be a tight bound to the capacity of the relay channel in general.

To settle this conjecture, it should be studied in future research whether Theorem 1 can be extended to the rank-deficient case in order to derive a method to find the globally optimal PDF rate without the approximation used in this paper.

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