Point Source Estimation in Sensor Networks using Compressed Sensing with Gaussian Kernels

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I. Introduction

Sensor networks play an important role in environmental monitoring, e.g., of water quality in water bodies, air pollution [1], or earthquake detection [2]; but also in monitoring of, e.g., chilled food transports [3]. These scenarios have in common that the source of the quantity to be measured, i.e., a pollutant, seismic event, or a heat source, can be assumed point-shaped. Additionally, the propagation of the quantity under measurement follows differential equations, which often represent a diffusion process. A spatially distributed sensor network is able to perform a scalar measurement of the quantity of interest at different positions. Now, the task of this sensor network is to make an estimation of the location of the sources and/or obtain an estimate of the spatial distribution of the quantity of interest, i.e., of its field. In contrast to previous works using Compressed Sensing (CS) techniques for the estimation of diffusion fields, e.g., [4], we will present a flexible framework able to use different basis functions and demonstrate its effectiveness using radial basis functions (RBFs), which are also known as Gaussian kernels.

Based on this framework, in this extended abstract, we will present a practical approach to perform an estimation for both location and field, based on the measurements of a sensor network. Due to space limitations, we will here restrict to the system model and the results for a centralized estimation in a data fusion center. In the final paper, we will also address the task of distributed estimation within the network of sensors (In-Network Processing).

II. System Model

The solution of any differential equation can be expressed using its specific so-called Green's function, which can be interpreted as the spatio-temporal impulse response of the system. For a one-dimensional diffusion process without boundaries, it reads [5]:

$$G(\mathbf{x}, t) = \frac{1}{\sqrt{4\pi k t}} \mathrm{e}^{-\frac{||\mathbf{x}||^2}{4kt}}, \tag{1}$$

with diffusion constant k, Cartesian coordinate $\mathbf{x}$ and time $t > 0$. If the M sources are assumed to have a point nature, they can be modeled as spatial delta functions with a temporal envelope $a_m(t)$:

$$s_m(\mathbf{x}, t) = a_m(t) \cdot \delta_0(\mathbf{x} - \mathbf{x}_m), \quad m = 1 \dots M. \tag{2}$$

The entire field $f(\mathbf{x}, t)$ can then be obtained by superimposing the spatio-temporal convolution of each of the sources with the Green's function:

$$f(\mathbf{x}, t) = \sum_{m=1}^{M} s_m(\mathbf{x}, t) \overset{\mathbf{x},t}{*} G(\mathbf{x}, t). \tag{3}$$

Taking into account the sifting property of the delta function, only the temporal convolution remains:

$$f(\mathbf{x}, t) = \sum_{m=1}^{M} a_m(t) \overset{t}{*} G(\mathbf{x} - \mathbf{x}_m, t). \tag{4}$$

If furthermore the sources are assumed to be activated at $t = 0$ and to maintain the same amplitude a_m afterwards, the convolution integral can be expressed as

$$f(\mathbf{x}, t) = \sum_{m=1}^{M} a_m \int_0^t G(\mathbf{x} - \mathbf{x}_m, \tau) d\tau. \tag{5}$$

This integral evaluates to

$$\int_0^t G(\mathbf{x} - \mathbf{x}_m, \tau) d\tau = \sqrt{\frac{t}{\pi k}} \mathrm{e}^{-\frac{||\mathbf{x}-\mathbf{x}_m||^2}{4kt}} + \frac{||\mathbf{x} - \mathbf{x}_m||}{2k} \mathrm{erfc}\left(\frac{||\mathbf{x} - \mathbf{x}_m||}{2\sqrt{kt}}\right), \tag{6}$$

where $\mathrm{erfc}(\cdot)$ represents the error function complement. For this extended abstract we will restrict to the single time instant $t = T$ and furthermore omit the second term, i.e., restrict our investigations to a purely Gaussian kernel[1]. Therefore, the true

[1] The authors are well aware that this will result in an approximation error, in particular for the higher dimensional case, where the Green's function's time dependency is different and the convolution integral (5) therefore yields different results. E.g., for a two-dimensional diffusion process, an exponential integral function is obtained for (6), which differs significantly from the Gaussian kernel. However, the results shown later justify this approach. The general suitability of different basis functions for the purpose of approximating measured fields has been investigated in [6].

field $f(\mathbf{x}, t)$ is approximated by

$$\tilde{f}(\mathbf{x}, t) = \sum_{m=1}^{M} \underbrace{a_m \sqrt{\frac{T}{\pi k}}}_{\tilde{a}_m} \mathrm{e}^{-\frac{||\mathbf{x}-\mathbf{x}_m||^2}{4kT}}. \tag{7}$$

Using (7), every measurement y_j of the sensor $j = 1, \ldots, J$ located at position $\mathbf{x}_j$ can now be described by

$$y_j = \sum_{m=1}^{M} \tilde{a}_m \mathrm{e}^{-\frac{||\mathbf{x}_j-\mathbf{x}_m||^2}{4kT}} + n_j, \tag{8}$$

where n_j accounts for any measurement error.

Please note that this kind of modeling as also very common in the field of kernel adaptive filtering, e.g., [7]. Also, the Kriging method [3], [8] uses a modified Gaussian kernel to approximate the so-called variogram, i.e., the effect of sources on the measurements.

The philosophy of [9] is generalized to a two-dimensional grid, i.e., the possible source locations are quantized to $N \cdot L$ possible coordinates and follow the form $\mathbf{x}_m = (n_m\Delta_1, \ell_m\Delta_2) + \mathbf{x}_0$, with $n_m = 0, \ldots, N-1$, $\ell_m = 0, \ldots, L-1$ and $\mathbf{x}_0$ an arbitrary offset. This offset, as well as N, L, Δ_1 and Δ_2, needs to chosen properly and according to the scenario.

Every sensor measurement y_j is now interpreted as superposition of the effect of $N \cdot L$ hypothetical sources with amplitudes $\hat{a}_m$, $m = 1, \ldots, NL$:

$$y_j = \sum_{m=1}^{N\cdot L} \hat{a}_m \mathrm{e}^{-\frac{||\mathbf{x}_j-\mathbf{x}_m||^2}{4kT}}. \tag{9}$$

Combining these equations for all sensors $j = 1, \ldots, J$, the equation system

$$\mathbf{y} = \mathbf{\Phi}\hat{\mathbf{a}} \tag{10}$$

is obtained, with

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_J \end{bmatrix}, \quad \hat{\mathbf{a}} = \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \vdots \\ \hat{a}_{N\cdot L} \end{bmatrix}, \tag{11}$$

and measurement matrix

$$\mathbf{\Phi} = \begin{bmatrix} \mathrm{e}^{-\frac{||\mathbf{x}_1-\mathbf{x}_1||^2}{4kT}} & \mathrm{e}^{-\frac{||\mathbf{x}_1-\mathbf{x}_2||^2}{4kT}} & \ldots & \mathrm{e}^{-\frac{||\mathbf{x}_1-\mathbf{x}_{N\cdot L}||^2}{4kT}} \\ \mathrm{e}^{-\frac{||\mathbf{x}_2-\mathbf{x}_1||^2}{4kT}} & \mathrm{e}^{-\frac{||\mathbf{x}_2-\mathbf{x}_2||^2}{4kT}} & \ldots & \mathrm{e}^{-\frac{||\mathbf{x}_2-\mathbf{x}_{N\cdot L}||^2}{4kT}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{e}^{-\frac{||\mathbf{x}_J-\mathbf{x}_1||^2}{4kT}} & \mathrm{e}^{-\frac{||\mathbf{x}_J-\mathbf{x}_2||^2}{4kT}} & \ldots & \mathrm{e}^{-\frac{||\mathbf{x}_J-\mathbf{x}_{N\cdot L}||^2}{4kT}} \end{bmatrix}. \tag{12}$$

(10) can be solved for $\hat{\mathbf{a}}$, e.g., by a least squares approach[2]. This will ensure minimum error between measured values and reconstructed field at sensor location, but usually not result in a smooth interpolation between the sensors. Since the underlying physical process is assumed to only exhibit few sources, a sparse estimate for $\hat{\mathbf{a}}$ is expected to show a better performance in the sense of reproducing the original field.

[2]If the measurements y_j possess a common, but unknown offset, this can be accounted for by augmenting the vector $\hat{\mathbf{a}}$ by this offset coefficient and extending the matrix $\mathbf{\Phi}$ by an all-ones column.

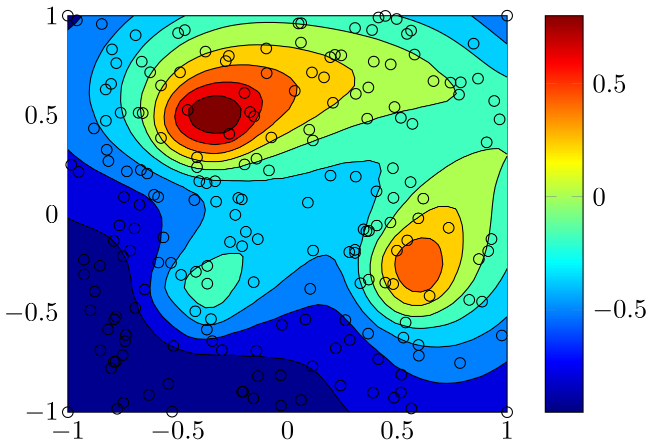

Fig. 1. Field to be estimated with sensor locations denoted by circles

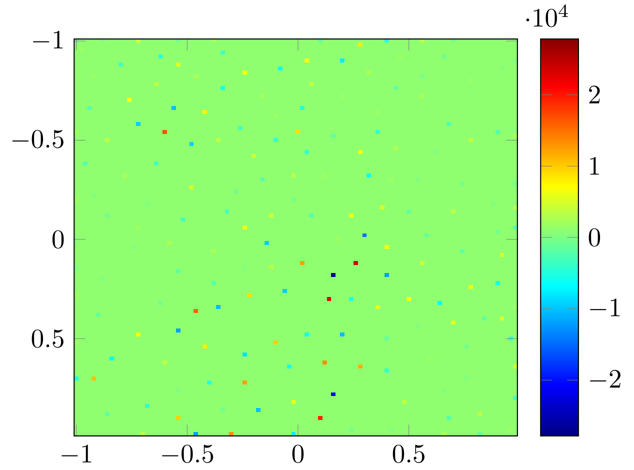

Fig. 2. Estimated coefficients $\hat{\mathbf{a}}$ for the LS criterion

III. Preliminary Results

Fig. 1 shows an exemplary field with 3 disk shaped sources and a combined diffusion and convection process that has been generated using the COMSOL CFD software [6]. This field is sampled by $J = 200$ sensors placed randomly at the positions denoted by circles. One possible least squares solution of the underdetermined equation system (10) for $N = L = 100$, $T = 1$ and $k = 0.01$ yields the coefficients $\hat{\mathbf{a}}$ as shown in Fig. 2. Here, the solution with the smallest number of nonzero elements was chosen. The reconstructed field according to (7) is shown in Fig. 3. It obviously shows no resemblance to the original field, which also can be quantified by means of the mean square error (MSE) between original field and estimate, which has been obtained by comparing both fields on the same 100×100 grid also used for estimation. For the LS criterion, an MSE of 1.26×10^4 is obtained.

If the estimation of $\hat{\mathbf{a}}$ is performed in a Compressed Sensing fashion, i.e., its sparsity is enforced within the estimation

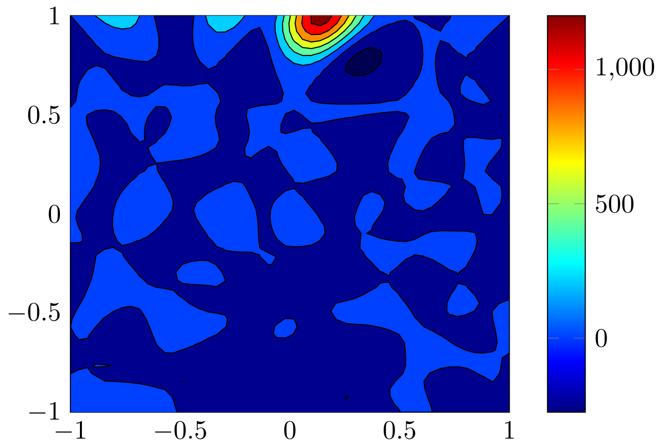

Fig. 3. Estimated field using the LS criterion

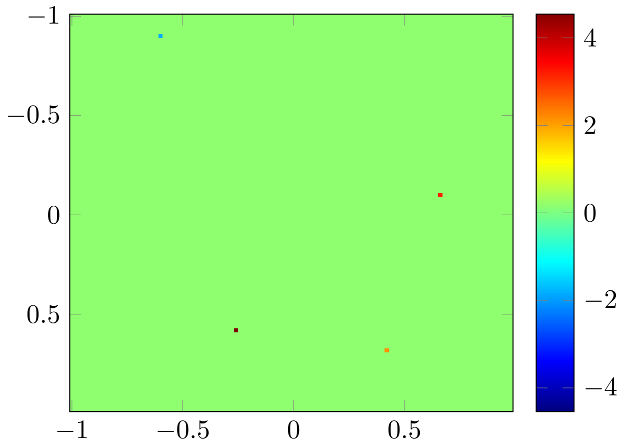

Fig. 4. Estimated coefficients $\hat{\mathbf{a}}$ using the OMP algorithm for a sparsity of 4

criterion, a better recontruction of the field can be achieved. In this extended abstract we will present results obtained using the Orthogonal Matching Pursuit (OMP) [10] algorithm. Fig. 4 shows the coefficients $\hat{\mathbf{a}}$ obtained by the OMP algorithm with a prescribed sparsity of 4. The resulting recontructed field is shown in Fig. 5. This field follows the shape of the original field in a better way, which also is reflected in a much smaller MSE of approx. 0.48.

IV. Outlook

In the final paper we also present results for the LASSO algorithm and other basis functions such as splines [6]. furthermore, we will show how the proposed algorithm can be distributed onto nodes of a wireless sensor network in order to achieve a decentralized estimation.

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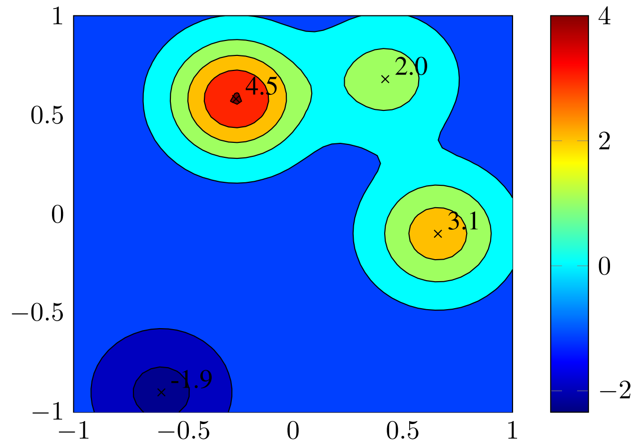

Fig. 5. Estimated field using the OMP algorithm for a sparsity of 4 with estimated point sources indicated by crosses and their amplitudes

References

[1] J. Ranieri, I. Dokmanic, A. Chebira, and M. Vetterli, "Sampling and reconstruction of time-varying atmospheric emissions," in *IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, March 2012, pp. 3673–3676.

[2] R. Tan, G. Xing, J. Chen, W.-Z. Song, and R. Huang, "Quality-driven volcanic earthquake detection using wireless sensor networks," in *IEEE 31st Real-Time Systems Symposium (RTSS)*, Nov 2010, pp. 271–280.

[3] R. Jedermann and W. Lang, "The minimum number of sensors - interpolation of spatial temperature profiles in chilled transports," in *Wireless Sensor Networks*, ser. Lecture Notes in Computer Science, U. Roedig and C. J. Sreenan, Eds. Springer Berlin Heidelberg, 2009, vol. 5432, pp. 232–246.

[4] M. Rostami, N.-M. Cheung, and T. Quek, "Compressed sensing of diffusion fields under heat equation constraint," in *IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, May 2013, pp. 4271–4274.

[5] J. Ranieri and M. Vetterli, "Sampling and reconstructing diffusion fields in presence of aliasing," in *2013 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, May 2013, pp. 5474–5478.

[6] R. Jedermann and H. Paul, "Accuracy of spatial field reconstruction in wireless sensor networks - what can 100 sensors tell us?" *submitted to Hindawi International Journal of Distributed Sensor Networks*.

[7] S. v. Vaerenbergh, "Kernel methods for nonlinear identification, equalization and separation of signals," Ph.D. dissertation, Universidad de Cantabria.

[8] J. Chilès and P. Delfiner, *Geostatistics - modeling spatial uncertainty*. John Wiley & Sons, New York, 1999.

[9] J. Ranieri, A. Chebira, Y. Lu, and M. Vetterli, "Sampling and reconstructing diffusion fields with localized sources," in *IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, May 2011, pp. 4016–4019.

[10] Y. Pati, R. Rezaiifar, and P. Krishnaprasad, "Orthogonal matching pursuit: recursive function approximation with applications to wavelet decomposition," in *Conference Record of The Twenty-Seventh Asilomar Conference on Signals, Systems and Computers*, Nov 1993, pp. 40–44 vol.1.