On Physical Limits of Massive MISO Systems

Michel T. Ivrlač Josef A. Nossek

Institute for Circuit Theory and Signal Processing
Technische Universität München, D-80333 Munich, Germany
email: {ivrlac, nossek}@tum.de

***Abstract*—We analyze the performance of a fixed-size uniform circular array which is used to transmit information to a single antenna receiver as a function of the number of transmit side antennas. We find that the minimum necessary energy to transfer one information bit stops decreasing once the number of antennas grows over a certain bound. We relate this bound to the diameter of the array and show that there is an optimum and finite number of antennas for every such array. For large array diameters this optimum number grows linearly with the diameter, the factor of proportionality primarily depending on whether the receiver is inside or outside of the transmit uniform circular array.**

I. Introduction

Massive MIMO systems are currently considered a possible key technology for the next generation of wireless communication systems [1]. The idea is that the number of antennas at the base station is much larger than the total number of antennas of the served user terminals. This may require hundreds or even thousand of antennas at the base station.

There are a number of possible advantages of such an approach. For example, Russek et al. point out in [2] that massive MIMO systems 1) allow linear signal processing techniques to reach near optimum performance, 2) provide a natural stage for improved analysis based on random matrix theory [3], and 3) allow that thermal noise can be averaged out since coherent averaging offered by a receive antenna array would eliminate quantities that are uncorrelated between the antenna elements, and especially thermal noise. In [1] Larsson et al. additionally point out that 4) if an antenna array were serving a single terminal, then it could be shown that the total necessary transmit power could be made inversely proportional to the number of antennas at the transmitter.

While assertions 1) and 2) above might ring true, the assertions 3) and 4) look problematic. Since increasing the number of antennas in a fixed space requires the average antenna separation to decrease, the inevitable electromagnetic interaction of the antennas leads to correlated instead of uncorrelated thermal noise, which violates the basic assumption of assertion 3). Similarly, electromagnetic interaction leads to the effect that the power which is radiated by an antenna array is not proportional to the sum of squares of the antenna's excitation [4]. A consequence of this is that, when the number of antennas grows beyond a certain bound, the radiated power to ensure a preset signal quality at the receiver does not drop any more by adding still more antennas. This is the subject of this paper.

II. System Model

Figure 1 schematically shows the system under investigation. It consists of a uniform circular array (UCA) of N quarter wavelength monopoles used for transmission, and one single

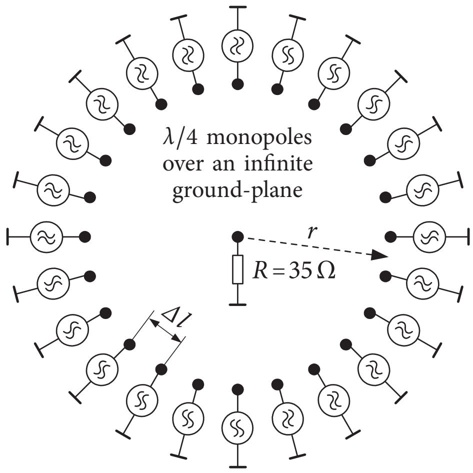

Figure 1: Uniform circular array transmits to a single antenna receiver located in the formers center.

such monopole, located in the center of the circle, used for reception. The center monopole is loaded with a resistance of $R = 35\,\Omega$, while the N UCA monopoles are fed by linear generators with the same output impedance of R. All monopoles reside over an infinite groundplane in an otherwise empty half-space, while the half-space below the groundplane contains the generators and the termination resistance. Denoting with r the radius of the UCA, the distance between neighboring monopoles equals $\Delta l = 2r \sin \pi/N$. Keeping r constant, Δl must decrease with increasing N towards zero. This ever increasing proximity creates strong mutual electromagnetic interaction between all antennas and has to be modeled carefully. To this end, we set up and analyze a linear muliport model for the $(N+1)$ antenna system which tells us 1) how the output voltage across the load resistor of the center monopole depends on the N open-circuit voltages of the linear generators, 2) how the radiated power depends on the generators' open circuit voltages, and 3) how large the variance of the noise voltage in a certain bandwidth is.

A. Input-Output Relationship

Let us first imagine that there is no noise. Figure 2 then shows the linear multiport model of the system under investigation. Let the $(N+1)$ port be described by

$$\begin{bmatrix} \boldsymbol{u} \\ u \end{bmatrix} = \begin{bmatrix} \boldsymbol{Z}_\mathrm{T} & \boldsymbol{z} \\ \boldsymbol{z}^\mathrm{T} & Z_\mathrm{R} \end{bmatrix} \begin{bmatrix} \boldsymbol{i} \\ i \end{bmatrix}, \tag{1}$$

where $\boldsymbol{u}$ and $\boldsymbol{i}$ are the N-dimensional vectors of the complex envelopes of the port voltages and port currents of the N excitation ports of the UCA's antennas, respectively, while u

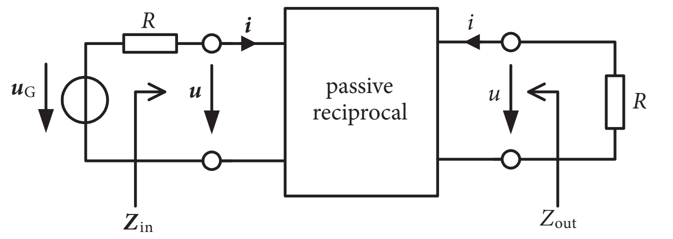

Figure 2: Linear multiport model of the considered system for the noise-free case.

and i are the complex envelopes of the port voltage and current at the center monopole's excitation port. The impedance matrix of the multiport is composed of the $N \times N$ transmit-side impedance matrix $\boldsymbol{Z}_{\mathrm{T}}$, the receive-side impedance Z_{R}, and the $N \times 1$ trans-impedance vector $\boldsymbol{z}$ which contains the mutual impedances between every of the UCA's antennas and the center monopole. The multiport is passive and reciprocal, which means that $\boldsymbol{Z}_{\mathrm{T}} = \boldsymbol{Z}_{\mathrm{T}}^{\mathrm{T}}$ and that the real-part of the whole $(N+1) \times (N+1)$ impedance matrix is non-negative definite. The N left-hand-side ports are connected to N linear generators with internal impedance $R > 0$, which complex envelopes of their open-circuit voltages are collected into the $N \times 1$ vector $\boldsymbol{u}_{\mathrm{G}}$. Finally, the remaining port is terminated with the passive resistance R across which the output voltage with the complex envelope u appears. Because of the linearity of the system, the relationship between u and $\boldsymbol{u}_{\mathrm{G}}$ can be written as

$$u = \boldsymbol{g}^{\mathrm{T}} \boldsymbol{u}_{\mathrm{G}}, \tag{2}$$

where the vector $\boldsymbol{g}^{\mathrm{T}}$ is obtained from basic circuit analysis:

$$\boldsymbol{g}^{\mathrm{T}} = \frac{R}{R + Z_{\mathrm{R}}} \boldsymbol{z}^{\mathrm{T}} \left(\boldsymbol{Z}_{\mathrm{in}} + R\mathbf{I} \right)^{-1}, \tag{3}$$

where $\boldsymbol{Z}_{\mathrm{in}}$ is the input impedance of the multiport seen from the transmit side:

$$\boldsymbol{Z}_{\mathrm{in}} = \boldsymbol{Z}_{\mathrm{T}} - \frac{\boldsymbol{z}\boldsymbol{z}^{\mathrm{T}}}{R + Z_{\mathrm{R}}}. \tag{4}$$

B. Transmit and Receive Power

The total active power P_{T}, which is delivered by the generators equals $P_{\mathrm{T}} = \mathrm{E}\left[\mathrm{Re}\{\boldsymbol{u}^{\mathrm{H}} \boldsymbol{i}\}\right]$. It can also be written as:

$$P_{\mathrm{T}} = \frac{1}{4R} \mathrm{E}\left[\boldsymbol{u}_{\mathrm{G}}^{\mathrm{H}} \boldsymbol{B} \boldsymbol{u}_{\mathrm{G}}\right], \tag{5}$$

where

$$\boldsymbol{B} = 4R \left(\boldsymbol{Z}_{\mathrm{in}} + R\mathbf{I} \right)^{-\mathrm{H}} \mathrm{Re}\{\boldsymbol{Z}_{\mathrm{in}}\} \left(\boldsymbol{Z}_{\mathrm{in}} + R\mathbf{I} \right)^{-1}, \tag{6}$$

and $\mathrm{E}[\cdot]$ refers to the expectation operation while the superscript $^{\mathrm{H}}$ denotes the complex conjugate transpose. If the antennas are lossless, the power P_{T} is radiated completely, otherwise part of it is dissipated into antenna heat-loss. On any rate, we call P_{T} the *transmit power.* The received signal power equals the power which is dissipated in the load resistor R, and can be obtained from:

$$P_{\mathrm{R}} = \mathrm{E}\left[|u|^2\right] / R. \tag{7}$$

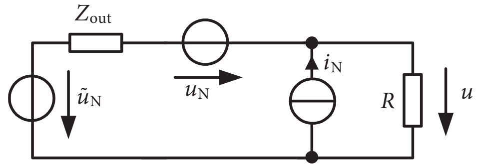

Figure 3: Equivalent circuit for determining the noise voltage.

C. Output Noise Voltage

Let us now consider the output voltage which is due to both the noise which is received by the center monopole and which is due to the noise of the receiver's low noise amplifier (LNA). Figure 3 shows the equivalent circuit which we can use to find the variance of the output noise voltage. The center monopole is represented by a voltage source with impedance

$$Z_{\mathrm{out}} = Z_{\mathrm{R}} - \boldsymbol{z}^{\mathrm{T}} \left(\boldsymbol{Z}_{\mathrm{T}} + R\mathbf{I} \right)^{-1} \boldsymbol{z}, \tag{8}$$

and an open-circuit noise voltage with complex envelope $\tilde{u}_{\mathrm{N}}$, which is characterized by [5]

$$\mathrm{E}\left[|\tilde{u}_{\mathrm{N}}|^2\right] = 4\mathrm{k}_{\mathrm{B}} T W \, \mathrm{Re}\{Z_{\mathrm{out}}\}, \tag{9}$$

where k_{B} is Boltzmann's constant, W is the (small) noise bandwidth, and T is the antenna noise temperature. The noise of the LNA is taken into account by the voltage noise source u_{N} and the current noise source i_{N}, which statistical properties depend on the LNA. To fix ideas, we assume the noise properties of the LNA to be specified as:

$$\mathrm{E}\left[|u_{\mathrm{N}}|^2\right] = 2\mathrm{k}_{\mathrm{B}} T W R, \quad \mathrm{E}\left[|i_{\mathrm{N}}|^2\right] = 2\mathrm{k}_{\mathrm{B}} T W / R, \quad \mathrm{E}\left[u_{\mathrm{N}} i_{\mathrm{N}}^*\right] = 0. \tag{10}$$

From this follows that $Z_{\mathrm{opt}} = R$, and $\mathrm{NF}_{\min} = 2$, which means that our LNA shall have minimum noise figure of 3 dB, which it achieves if it is driven with a source of impedance equal to R (see e.g., [6] for details). Moreover, we assume that LNA noise and antenna noise are *uncorrelated.* From the circuit in Figure 3, it is clear that

$$u = \frac{R}{R + Z_{\mathrm{out}}} \left(\tilde{u}_{\mathrm{N}} - u_{\mathrm{N}} + Z_{\mathrm{out}} i_{\mathrm{N}} \right).$$

With $\tilde{u}_{\mathrm{N}}$ uncorrelated with u_{N} and i_{N}, it then follows with the help of (9) and (10), that

$$\mathrm{E}\left[|u|^2\right] = 2\mathrm{k}_{\mathrm{B}} T W R. \tag{11}$$

D. Linear System Model

Because of linearity, the total output voltage is the sum of the deterministic output voltage as given in (2) and the noise voltage with variance given in (11). Thus, the complete linear system model can be written as

$$\begin{aligned} u &= \boldsymbol{g}^{\mathrm{T}} \boldsymbol{u}_{\mathrm{G}} + \eta, \quad \eta \sim \mathcal{CN}\left(0, 2\mathrm{k}_{\mathrm{B}} T W R\right), \\ P_{\mathrm{R}} &= \mathrm{E}\left[|u|^2\right] / R\big|_{\eta = 0} \quad P_{\mathrm{T}} = \tfrac{1}{4R} \mathrm{E}\left[\boldsymbol{u}_{\mathrm{G}}^{\mathrm{H}} \boldsymbol{B} \boldsymbol{u}_{\mathrm{G}}\right]. \end{aligned} \tag{12}$$

To ease further work, we now introduce two bijective linear transformations:

$$\boldsymbol{x} = \frac{1}{2\sqrt{R}} \boldsymbol{B}^{1/2} \boldsymbol{u}_{\mathrm{G}}, \quad y = u / \sqrt{R}. \tag{13}$$

Note from (6) and (4), that $\boldsymbol{B}=\boldsymbol{B}^{\mathrm{H}}$, because $\boldsymbol{Z}_{\mathrm{T}}=\boldsymbol{Z}_{\mathrm{T}}^{\mathrm{T}}$ due to antenna reciprocity. So $\boldsymbol{B}$ has got real eigenvalues and orthonormal eigenvectors. Since furthermore the antenna multiport is passive, we have that $\boldsymbol{B}>\boldsymbol{0}$, for the transmit power must be positive for any vector $\boldsymbol{u}_{\mathrm{G}} \neq \boldsymbol{0}$. This means that $\boldsymbol{B}$ has got positive eigenvalues. Thus, $\boldsymbol{B}^{1/2}$ has got *real-valued* eigenvalues and, consequently, $\boldsymbol{B}^{1/2}=\boldsymbol{B}^{\mathrm{H}/2}$. Solving the left hand equation of (13) for $\boldsymbol{u}_{\mathrm{G}}$ and substituting into the lower right-most term of (12), it follows that $P_{\mathrm{T}}=\mathrm{E}\left[\,\|\boldsymbol{x}\|_2^2\right]$. With the help of (13), we obtain from the upper left-most term of (12):

$$y = 2\boldsymbol{g}^{\mathrm{T}}\boldsymbol{B}^{-1/2}\boldsymbol{x} + \eta/\sqrt{R},$$

so that we finally obtain a linear system model which is equivalent to the one from (12):

$$\begin{aligned} y &= \boldsymbol{h}^{\mathrm{T}}\boldsymbol{x} + \vartheta, \quad \vartheta \sim \mathcal{CN}\left(0, 2k_{\mathrm{B}}TW\right), \\ P_{\mathrm{R}} &= \mathrm{E}\left[|y|^2\right]\big|_{\vartheta=0}, \quad P_{\mathrm{T}} = \mathrm{E}\left[\|\boldsymbol{x}\|_2^2\right], \end{aligned} \tag{14}$$

where the *channel vector* $\boldsymbol{h}^{\mathrm{T}}$ is given by

$$\boldsymbol{h}^{\mathrm{T}} = 2\boldsymbol{g}^{\mathrm{T}}\boldsymbol{B}^{-1/2}. \tag{15}$$

All the relevant physical properties of the multi-antenna system are now captured by the channel vector $\boldsymbol{h}^{\mathrm{T}}$.

III. Optimum Beamforming

Transmitting a signal s over the N antennas of the UCA such that it can be received by the center monopole with the largest possible signal to noise ratio (SNR), we apply beamforming:

$$\boldsymbol{x} = \boldsymbol{t}s, \tag{16}$$

where $\boldsymbol{t}$ is the beamforming vector. The maximum SNR then becomes with the help of (14):

$$\mathrm{SNR} = \max_{\boldsymbol{t}} \frac{\mathrm{E}\left[|y|^2|\vartheta=0\right]}{\mathrm{E}\left[|y|^2|s=0\right]} = \max_{\boldsymbol{t}} \frac{P_{\mathrm{T}}}{2k_{\mathrm{B}}TW}\frac{\left|\boldsymbol{h}^{\mathrm{T}}\boldsymbol{t}\right|^2}{\|\boldsymbol{t}\|_2^2} = \frac{P_{\mathrm{T}}\,\|\boldsymbol{h}\|_2^2}{2k_{\mathrm{B}}TW}, \tag{17}$$

where the last equality is a consequence of the Cauchy-Schwarz inequality. The optimum beamforming vector is proportional to $\boldsymbol{h}^*$. Note that

$$\max \frac{P_{\mathrm{R}}}{P_{\mathrm{T}}} = \|\boldsymbol{h}\|_2^2 \leq 1. \tag{18}$$

The inequality comes from the energy law such that we shall not receive more signal power than we have transmitted. Consequently, even though the number N of components of the channel vector can grow unboundedly, its Euclidean norm can never exceed unity.

IV. Energy per Information Bit

The system described in (14) with (16) is a complex additive white Gaußian noise (AWGN) channel, which channel capacity (maximum mutual information between s and y, where maximization is done over the probability density function of s) is given by [7]:

$$C = W\log_2\left(1+\mathrm{SNR}\right). \tag{19}$$

With C information bits per second, it takes the time $T_{\mathrm{b}}=1/C$ to transfer a single information bit. The energy necessary to

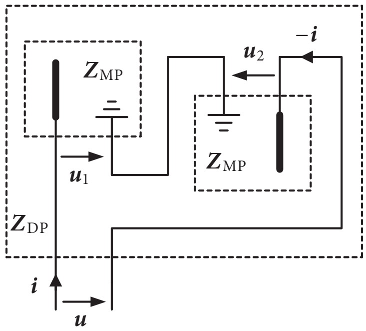

Figure 4: Multiport model of a dipole-array as a concatenation of two monopole arrays which are separated by an infinite ground plane.

accomplish this transfer is, therefore, $E_{\mathrm{b}}^{(C)}=P_{\mathrm{T}}T_{\mathrm{b}}=P_{\mathrm{T}}/C$. Solving (19) for SNR, substituting the result into (17) and solving the latter for P_{T}, it therefore follows that

$$E_{\mathrm{b}}^{(C)} = \frac{2k_{\mathrm{B}}TW}{\|\boldsymbol{h}\|_2^2}\cdot\frac{2^{C/W}-1}{C}. \tag{20}$$

Let E_{b} be the *smallest* value of $E_{\mathrm{b}}^{(C)}$ for a given $\|\boldsymbol{h}\|_2^2$, namely

$$E_{\mathrm{b}} = \min_C E_{\mathrm{b}}^{(C)}. \tag{21}$$

This minimum is obtained for $C\to 0$ and results in:

$$\frac{E_{\mathrm{b}}}{k_{\mathrm{B}}T} = \frac{\log_{\mathrm{e}}4}{\|\boldsymbol{h}\|_2^2} \geq \log_{\mathrm{e}}4. \tag{22}$$

The minimum necessary energy per information bit can never drop below a well defined positive limit no matter how many antennas N are used at the transmitter. Substituting (15) into (22) it then follows with the help of (3), (4) and (6) that

$$\frac{E_{\mathrm{b}}}{k_{\mathrm{B}}T} = \frac{|R+Z_{\mathrm{R}}|^2}{\boldsymbol{z}^{\mathrm{H}}\left(\mathrm{Re}\left\{\boldsymbol{Z}_{\mathrm{T}}-\dfrac{\boldsymbol{z}\boldsymbol{z}^{\mathrm{T}}}{R+Z_{\mathrm{R}}}\right\}/R\right)^{-1}\boldsymbol{z}}\cdot\log_{\mathrm{e}}4, \tag{23}$$

V. The Impedance Matrix

Imagine two monopole arrays over an infinite ground plane (infinitely thin, infinitely large and infinitely well conducting), one of them rotated such that its monopoles point in the opposite direction as those of the other. By aligning the two arrays so that their ground planes touch and become one infinite ground plane with monopoles sticking out at each side symmetrically, we obtain an array of dipoles with an infinite ground plane. The multiport model for this construction is shown in Figure 4 and leads us to

$$\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2 = \boldsymbol{Z}_{\mathrm{MP}}\boldsymbol{i} - \boldsymbol{Z}_{\mathrm{MP}}\left(-\boldsymbol{i}\right) = 2\boldsymbol{Z}_{\mathrm{MP}}\boldsymbol{i},$$

where $\boldsymbol{Z}_{\mathrm{MP}}$ is the impedance matrix of the monopole array over an infinite ground plane. From $\boldsymbol{u}=\boldsymbol{Z}_{\mathrm{DP}}\boldsymbol{i}$, we can then relate the impedance matrix $\boldsymbol{Z}_{\mathrm{DP}}$ of a dipole array with infinite ground plane to the impedance matrix $\boldsymbol{Z}_{\mathrm{MP}}$ of the respective monopole array over an infinite ground plane:

$$\boldsymbol{Z}_{\mathrm{MP}} = \frac{1}{2}\boldsymbol{Z}_{\mathrm{DP}}. \tag{24}$$

The infinitely thin ground plane has no effect on the dipole array. This is because of its symmetry, which demands that no tangential components of the electric field are produced inside the ground plane by the currents that flow in the dipole wires. Therefore, no currents have to flow inside the ground plane to counter any tangential electric field components (as there aren't any). Therefore, no field is produced by the (infinitely thin) ground plane in case of the dipole array. Therfore, $\boldsymbol{Z}_{\mathrm{DP}}$ is the same as that of a dipole array without the ground plane. If one further assumes that the wires are infinitely thin, yet still perfectly conducting, then classical antenna theory provides a closed-form solution for all components of $\boldsymbol{Z}_{\mathrm{DP}}$. From the equations (13-23), (13-24), and (13-25) in [8], one obtains with the help of (24) for the mutual impedance

$$Z_{i,j} = R_{i,j} + \mathrm{j}X_{i,j}, \tag{25}$$

between any pair of two infinitely thin monopoles with length

$$l = \lambda/4, \tag{26}$$

with λ denoting the wavelength, the expressions

$$R_{i,j} = \frac{2\mathrm{Ci}(2\pi d_{i,j}/\lambda) - \mathrm{Ci}(\zeta_{i,j}+\pi) - \mathrm{Ci}(\zeta_{i,j}-\pi)}{8\pi\epsilon_0 c}, \tag{27}$$

and

$$X_{i,j} = \frac{-2\mathrm{Si}(2\pi d_{i,j}/\lambda) + \mathrm{Si}(\zeta_{i,j}+\pi) + \mathrm{Si}(\zeta_{i,j}-\pi)}{8\pi\epsilon_0 c}, \tag{28}$$

where

$$\zeta_{i,j} = \pi\sqrt{1 + 4d_{i,j}^2/\lambda^2}, \tag{29}$$

and $d_{i,j}$ is the distance between the monopole pair, while the Ci and Si denote the integral cosine and integral sine functions, respectively, and ϵ_0 is the electric constant, while c denotes the vacuum speed of light. The input impedance

$$Z_{i,i} = R + \mathrm{j}X = \lim_{d_{i,j}\to 0} Z_{i,j} \tag{30}$$

can also be obtained in closed form:

$$R = \frac{\Gamma - \mathrm{Ci}(2\pi) + \log 2\pi}{8\pi\epsilon_0 c}, \qquad X = \frac{\mathrm{Si}(2\pi)}{8\pi\epsilon_0 c}, \tag{31}$$

where $\mathit{\Gamma}$ is the Euler constant. Numerical evaluation yields

$$Z_{i,i} \approx (36.5 + \mathrm{j}21.3)\ \Omega.$$

In the UCA from Figure 1, the monopoles' feeding points are located at the Cartesian coordinates

$$x_i = r\cos\frac{2\pi i}{N}, \qquad y_i = r\sin\frac{2\pi i}{N},$$

where r is the radius of the UCA, and i ranges over all the integers from 1 to N. The distance between the i-th and j-th monopole is therefore given by

$$d_{i,j} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} = 2r\sin\frac{\pi|i-j|}{N}. \tag{32}$$

Finally, the distance $d_{i,0}$ between the i-th monopole of the UCA and the single receiving monopole equals

$$d_{i,0} = \sqrt{\left(x_0 - r\cos\frac{2\pi i}{N}\right)^2 + \left(y_0 - r\sin\frac{2\pi i}{N}\right)^2}, \tag{33}$$

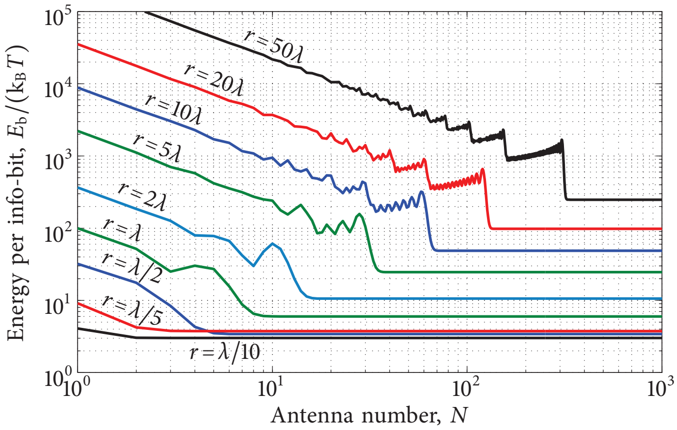

Figure 5: Energy per information bit as function of the number N of antennas for the scenario of Figure 1.

where x_0 and y_0 are the coordinates of the receiver. In Figure 1, we have $x_0 = y_0 = 0$. The impedance parameters in (1) can now be specified:

$$\left(\boldsymbol{Z}_{\mathrm{T}}\right)_{i,j} = Z_{i,j}, \quad Z_{\mathrm{R}} = Z_{1,1}, \quad \left(\boldsymbol{z}\right)_i = Z_{i,0} \tag{34}$$

where $i, j \in \{1, 2, \ldots, N\}$.

VI. Receiver in the Array Center

Let us have a look at the case shown in Figure 1, where the receiver is located right in the center of the UCA. We analyze the minimum necessary energy to transfer one information bit by first setting up the impedance parameters according to (34), and then evaluating (23) numerically for a raising number N of UCA antennas. The value of the resistance R is set to

$$R = 35\,\Omega.$$

Figure 5 shows $E_{\mathrm{b}}/(k_{\mathrm{B}}T)$ as a function of the number N of UCA antennas for a number of different fixed radii ranging from very small arrays of $\lambda/10$ to rather large arrays of 50λ radius. Starting from a single antenna at the transmitter, we see that E_{b} first drops with increasing N, approximately reducing to half its value when N is increased twofold. This shows that the array gain increases roughly linearly with N when N is not too large. However, when N climbs over a certain (radius dependent) number (e.g. about 5 for $r = 10\lambda$) we observe a more and more irregular and non-monotonic behavior of E_{b} with respect to N. When another critical number of antennas is reached (e.g., about 60 for $r = 10\lambda$), E_{b} *sharply* decreases (e.g., by more than a factor of 5 for a 12% increase of the antenna number when $r = 10\lambda$). After this steep descent, E_{b} levels off almost immediately and remains at the same value (e.g., $48.74 k_{\mathrm{B}}T$ for $r = 10\lambda$), no matter how the antenna number is increased further. Thus, having a huge number of antennas available helps to decrease the necessary E_{b} up to some limit, where further increase of the antenna number has no effect. For instance, having more than 324 antennas on a circle of radius 50λ does not decrease E_{b} any further. For each radius, there is such an antenna number, N_{sat} say, for which further increase of N has no impact on E_{b} anymore. This N_{sat} is, therefore, the optimum antenna number which delivers (nearly) the

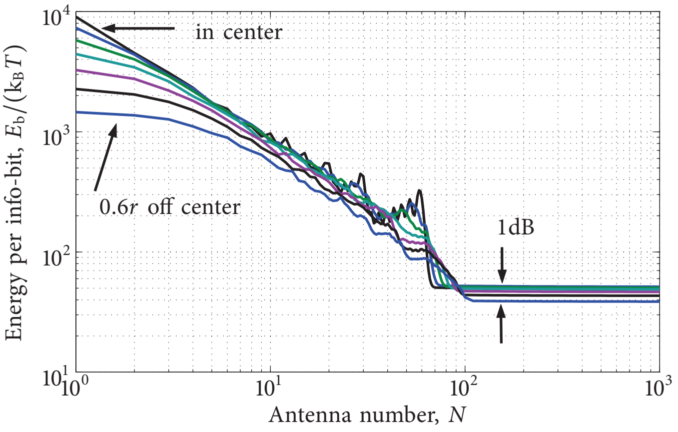

Figure 6: Energy per information bit as function of antenna number for 7 different receiver positions (x_0, y_0) with $y_0 = 0$ and $x_0 \in \{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6\} \cdot r$, where the radius r of the uniform circular array is set to $r = 10\lambda$.

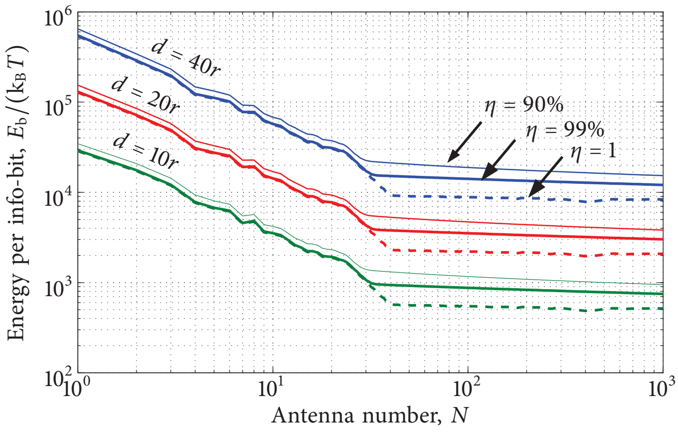

Figure 7: Energy per information bit as function of antenna number for 3 receiver distances (d), each of which puts the receiver outside the transmitter's circle of $r = 2\lambda$. Three radiation efficiencies (η) of the monopoles are considered.

maximum array gain

$$A_{\max} = \frac{E_b|_{N=1}}{\min_N E_b}. \tag{35}$$

From a close observation of Figure 5, it turns out that

$$\text{for } r \gg \lambda, \quad N_{\text{sat}} \approx \left\lfloor 2\pi \frac{r}{\lambda} \right\rfloor. \tag{36}$$

The optimum spacing Δl between adjacent monopoles in the UCA approaches one wavelength from below as r/λ grows towards infinity. This makes sense, because the receiver is positioned exactly in *front-fire* direction (see [4] for further information).

We note in passing that, for $r = \lambda/2$, the optimum antenna spacing is also $\Delta l = \lambda/2$, where one obtains an array gain of 9 from 6 antennas and an E_b which is 1.6 dB larger than the absolute minimum. Another interesting radius is 0.0675λ, because then exactly 50% of the transmitter power is received. While this system achieves only a meager array gain of 1.16 out of 2 antennas, the close proximity of receiver and transmitter allows for an E_b which comes closer than 0.7 dB to the absolute minimum of about $2.37 k_B T$.

VII. Off-Center Receiver

Let us now investigate what happens when we put the receiver away from the center of the UCA circle, in particular to the positions given by the following Cartesian coordinates:

$$y_0 = 0, \quad x_0 \in \{0,\ 0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6\} \cdot r.$$

We assume that optimum beamforming is performed for each position. In this way, we can still use formula (23) to determine the minimum required bit energy. The only change compared to having the receiver in the center is a different transimpedance vector $\boldsymbol{z}$, which reflects the different distances between each of the UCA's antennas and the receiver's antenna. Figure 6 shows the obtained results for an $r = 10\lambda$. For low antenna numbers, there is substantial difference in the minimum E_b for different positions of the receiver inside a circle of $0.6r$ radius. For $N = 3$ antennas, there is a difference of about 4 dB in the required bit energy depending on the position of the receiver. Even for N between about 10 and 40, the needed bit energy varies by about 2.5 dB. However, as N is increased until the bit energies saturate (in this case at about 110 antennas), the difference in E_b drops to 1 dB. That is, *inside a substantially large circle of 60% diameter of the UCA covering about 1/3 of the UCA's area, communication can take place with a bit energy within a range of* ±0.5 dB*, regardless of the position of the receiver within this circle, provided that enough antennas are used such that* E_b *saturates.* The necessary antenna number is about 40% larger than needed for centered receiver to obtain the energy saturation effect and now requires a spacing of about 0.57λ between neighboring antennas. The same spacing is also obtained for a larger circle of $r = 50\lambda$, as is the maximum difference in necessary E_b of ±0.5 dB. Thus, for large UCA radius, the minimum number of antennas needed to communicate to receivers in every position inside a circle of 60% radius around the center using bit energies differing no more than about 1 dB seems to be given in general by

$$N_{60\%} \approx \left\lfloor 11 \frac{r}{\lambda} \right\rfloor. \tag{37}$$

Note that optimum beamforming has to be performed for each receiver position, though.

VIII. Receiver Outside of Array

It is time now to look at the case where the receiver resides *outside* the uniform circular array. We consider the positions specified by the Cartesian coordinates:

$$y_0 = 0, \quad x \in \{10,\ 20,\ 40\} \cdot r,$$

for the case of $r = 2\lambda$. The results are shown in Figure 7. In contrast to the previous cases of the receiver being inside the UCA, optimum beamforming for receivers outside the array are somewhat sensitive to heat loss inside the antennas. We have therefore analyzed three different cases of antenna radiation efficiency, namely besides 100% also 99% and 90%, respectively. As can be seen in Figure 7, there is a region of antenna numbers where the energy is approximately inversely proportional to the antenna number. This region starts from $N = 1$ and extends to about $N = 40$ for lossless antennas and $N = 30$ in the two lossy cases. Further increase of the antenna number does not decrease the necessary bit energy significantly anymore. This is especially true for the case of lossless antennas. Interestingly, the lossy antennas show an improvement in bit energy even for much larger N. However, this improvement is rather slow, where a tenfold increase of the antenna number from 30 to 300 decreases the bit energy by merely 1.2 dB. For practical purposes, one can therefore consider the bit energy as being saturated already at $N = 30$. *This means that an approximately inverse relationship between antenna number and bit energy can be maintained until the distance between neighboring antennas drops below about* 0.42λ *for the lossy cases, and below about* 0.32λ *for the lossless case.* We conjecture therefore that the maximum useful antenna number in practice equals

$$N_{\max} \approx \left\lfloor 15\frac{r}{\lambda} \right\rfloor. \tag{38}$$

For $N \leq N_{\max}$, an antenna radiation efficiency of $\eta = 90\%$ demands an increase of bit energy by 0.9 dB compared to the lossless case. For $N > N_{\max}$, this increases is much larger and reaches up to 3.5 dB.

IX. The 3D Beamforming Effect

Suppose the receiver is located inside the UCA and optimum beamforming is made for the given receiver position (see (17)). It turns out that the required E_b is rather sensitive to changes in the receiver's position when the beamforming stays *fixed*. That is, for a fixed beamforming vector, there is a rather small area inside which the receiver can move while still enjoying a low E_b. In the following, the receiver moves from the center of the UCA along the positive x-axis, while the beamforming vector is held constant at

$$\boldsymbol{t} = \mathbf{1},$$

that is, the all-ones vector. Note that this coincides with the optimum beamforming vector only when the receiver is located right in the center of the UCA. For all other positions, the all-ones beamforming vector is strictly sub-optimum. The question we look at is how much worse it does compared to 1) optimum beamforming, and 2) to using only a single antenna at the transmitter. The single antenna is assumed to be located at the position $x_0 = r$, $y_0 = 0$. Setting $\boldsymbol{t} = \mathbf{1}$, the signal to noise ratio obtained changes from the one given in (17) to

$$\mathrm{SNR}_1 = \frac{P_T}{2k_B T W} \cdot \frac{1}{N} \left| \sum_{n=1}^{N} h_n \right|^2, \tag{39}$$

where h_n is the n-th component of the channel vector (15). Using this as the SNR in (19) we find that the minimum energy per bit for realiable communication is now given by

$$\frac{E_{b,1}}{k_B T} = \frac{N}{\left| \sum_{n=1}^{N} h_n \right|^2} \log_e 4. \tag{40}$$

The obtained results for a UCA radius of $r = 10\lambda$ are shown in Figure 8. We compare fixed (all-ones) beamforming to optimum beamforming for $N = 200$ antennas, which is in the region where the necessary bit energies have saturated, i.e., have become essentially independent of the antenna number. We assume that the radiation efficiency of the antennas equals $\eta = 99\%$ in all cases. Let us look at Figure 8 in some detail now. The prominent notch in E_b when the receiver is located at $(x, y) = (0, 10\lambda)$ comes about because the receiver gets very close to one of the UCA's antennas. In the center ($x = 0$) the all-ones beamforming is actually optimum. As long as the receiver is not removed far from the center, the needed bit energy therefore stays close to the minimum one. In fact, as long as the receiver is removed by less than $\lambda/9$ from the center, the loss in performance of the all-ones beamforming stays below 1 dB. For larger distances, however, the performance loss increases rapidly and soon becomes so bad that the all-ones beamforming performs worse than a single antenna transmitter. This occurs, for the first time, at a distance of about 0.36λ from the center. For distances greater than about 5.2λ, the performance of all-ones beamforming is consistently worse than that of the single antenna transmitter. Even the notch at position $x = 10\lambda$, while still present, is by far less deep than for optimum beamforming or for a single antenna. *The conclusion is that position dependent beamforming is absolutely necessary. An update of the beamforming vector should be performed no later than the receiver has traveled a distance of* $\lambda/9$ *in order to avoid excessive loss of performance.* For pedestrian speed and 30 GHz frequency, the update of the beamforming should be made at least every 10^{-3} seconds.

On the other hand, this strong sensitivity of E_b with user position for a fixed beamforming vector points to the possible application of separating two close-by receiver's. A separation of the receiver positions of the order of a wavelength might suffice to obtain a good spatial separation by beamforming. *This kind of 3D-beamforming could have the potential to drastically increase the number of servable receiver terminals per unit*

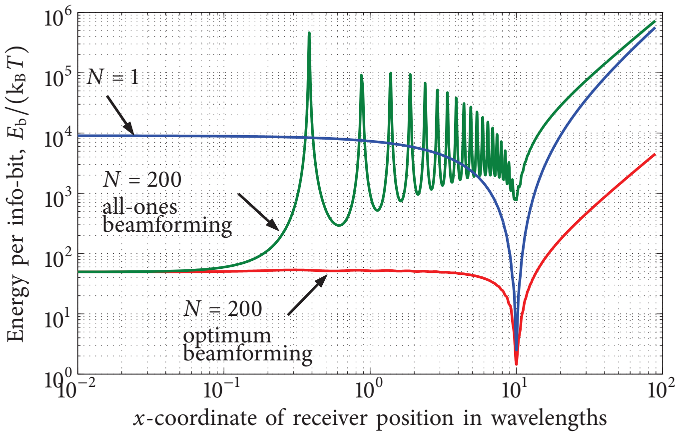

Figure 8: Energy per information bit as function of receiver position along the x-axis. Comparison of the optimum versus fixed (all-ones) beamforming for $N = 200$ antennas is shown. In addition, the energy needed when using a single antenna ($N = 1$) is plotted for reference. The radius of the uniform circular array is $r = 10\lambda$, while the radiation efficiency of the antennas is 99%.

of area. For this to work it appears essential that the receivers are surrounded *by sufficiently many antennas.* Further research in this area of 3D-beamforming with the receivers surrounded by many antennas seems advisable.

X. Conclusion

A theoretical study of a uniform circular array of quarter wavelength monopoles serving a receiver employing a single quarter wavelength monopole over an infinite ground plane has been carried out based on classical antenna theory and signal processing.

It is found that the energy to transfer one information bit decreases with the number of employed antennas at the transmitter up to some threshold at which a further increase of antenna number has no more effect on the required bit energy. The value of this threshold in antenna number depends on whether the receiver is surrounded by the transmitter's antennas, or is located well outside of the circular array. For a receiver right in the center of the array, the optimum antenna number is given such that the distance between neighboring antennas approaches one wavelength from below as the radius of the array becomes large with respect to the wavelength. For off-center receiver positions inside the array, the optimum distance is reduced to about 0.6λ, while for receivers well outside the array the optimum distance is about 0.4λ. Adaptive beamforming holds the necessary energy per information bit constant up to a peak deviation of $0.5\,\mathrm{dB}$ provided that 1) the receiver moves no more than about 60% of the radius from the center of the array, and 2) the array has got sufficiently many antennas (immediate neighbors separated by 0.6λ).

The beamforming is sensitive to the user position such that even a slight movement of a fraction of a wavelength requires an updated beamforming vector. While this may indicate limitations in speed of the receivers in a mobile environment, it also opens up the possibility to serve multiple, more or less static, users which are located very close to each other, separated by a distance of the order of the wavelength.

Such 3D-beamforming could potentially increase the density of servable receivers in a given area drastically. Further research in such beamforming techniques which surround the receiver by sufficiently many antennas seems an interesting research topic for future investigation by the public research community.

References

[1] E. Larsson, O. Edfors, F. Tufvesson, and T. Marzetta, "Massive MIMO for Next Generation Wireless Systems," *IEEE Communications Magazine*, vol. 52, no. 2, pp. 186–195, 2014.

[2] F. Russek, D. Persson, B. K. Lau, E. Larsson, T. Marzetta, O. Edfors, and F. Tufvesson, "Scaling Up MIMO: Opportunities and Challenges with Very Large Arrays," *IEEE Signal Processing Magazine*, vol. 30, no. 1, pp. 186–195, 2013.

[3] A. Tulino and S. Verdu, *Random Matrix Theory and Wireless Communications*. Now Publishers, 2004.

[4] M. Ivrlac and J. Nossek, "The Multiport Communication Theory," *IEEE Circuits and Systems Magazine*, vol. 14, no. 3, pp. 27–44, 2014.

[5] R. Q. Twiss, "Nyquist's and Thevenin's Theorems Generalized for Nonreciprocal Linear Networks," *Journal of Applied Physics*, vol. 26, pp. 599–602, May 1955.

[6] M. T. Ivrlač and J. A. Nossek, "Toward a Circuit Theory of Communication," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 57(7), pp. 1663–1683, 2010.

[7] C. E. Shannon, "A Mathematical Theory of Communications," *Bell Syst. Tech. Journal*, vol. 27, pp. 379–423, 623–656, jul 1948.

[8] S. A. Schelkunoff and H. T. Friis, *Antennas. Theory and Practice*. New York, NY: Wiley, 1952.